

## FOLIATION THEORY AND ITS APPLICATIONS\*

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ABSTRACT. Subject of present paper is the review of results of authors on foliation theory and applications of foliation theory in control systems. The paper consists of two parts. In the first part the results of authors on foliation theory are presented, in the second part the results on applications of foliation theory in the qualitative theory of control systems are given. In paper everywhere smoothness of a class  $C^\infty$  is considered.

Keywords: a Riemannian manifold, a foliation, a leaf, holonomy group, local stability theorem, level surfaces, metric function, total geodesic submanifold, connection, foliation with singularities, a orbit, control system, controllability set.

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### 1. INTRODUCTION

The foliation theory is a branch of the geometry which has arisen in the second half of the XX-th century on a joint of ordinary differential equations and the differential topology. Basic works on the foliation theory belong to the French mathematicians A. Haefliger [8], [9], G. Ehresman [5], [6], G. Reeb [45], [46], H. Rosenberg [48], [49], G. Lamoureaux [21], [22], R. Langevin [23], [24]. Important contribution to foliation theory was made by known mathematicians - as well as I. Tamura [53], R. Herman [10]-[13], T. Inaba [14], [15], W. Thurston [54], [55], P. Molino [26], P. Novikov [42], Ph. Tondeur [56], [57], B. Reinhart [47]. Now the foliation theory is intensively developed, has wide applications in various areas of mathematics - such, as the optimal control theory, the theory of dynamic polysystems. There are numerous researches on the foliation theory.

The review of the last scientific works on the foliation theory and very big bibliography is presented in work of Ph. Tondeur [57].

### 2. TOPOLOGY OF FOLIATIONS

**Definition 2.1.** Let  $(M, A)$  be a smooth manifold of dimension  $n$ , where  $A$  is a  $C^r$  atlas,  $r \geq 1$ ,  $0 < k < n$ . A family  $F = \{L_\alpha : \alpha \in B\}$  of path-wise connected subsets of  $M$  is called  $k$ -dimensional  $C^r$ -foliation of if it satisfies to the following three conditions:

$$F_I : \bigcup_{\alpha \in B} L_\alpha = M;$$

$$F_{II} : \text{for every } \alpha, \beta \in B \text{ if } \alpha \neq \beta, \text{ then } L_\alpha \cap L_\beta = \emptyset;$$

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$F_{III}$  : For any point  $p \in M$  there exists a local chart ( local coordinate system )  $(U, \varphi) \in A$ ,  $p \in U$  so that if  $U \cap L_\alpha \neq \emptyset$  for some  $\alpha \in B$  the components of  $\varphi(U \cap L_\alpha)$  are following subsets of parallel affine planes

$$\{(x_1, x_2, \dots, x_n) \in \varphi(U) : x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, \dots, x_n = c_n\}$$

where numbers  $c_{k+1}, c_{k+2}, \dots, c_n$  are constant on components (Figure-1,[53], p.121).

The most simple examples of a foliation are given by integral curves of a vector field and by level surfaces of differentiable functions. If the  $X$  vector field without singular points is given on manifold  $M$  under the theorem of straightening of a vector field (under the theorem of existence of the solution of the differential equation) integral curves generate one-dimensional foliation on  $M$ .

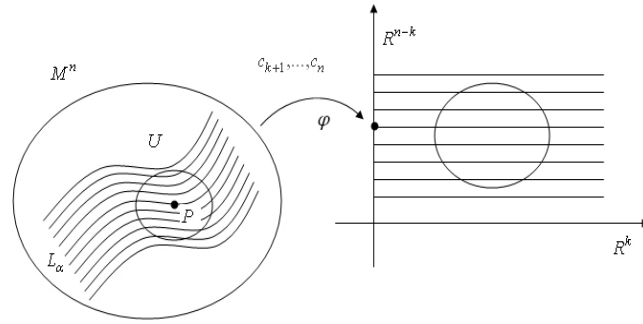


Figure 1

Let  $M$  be a smooth manifold of dimension  $n$ ,  $f : M \rightarrow R^1$  be a differentiable function. Let  $p_0 \in M, f(p_0) = c_0$  and the level set  $L = \{p \in M : f(p) = c_0\}$  does not contain critical points. Then the level set is a smooth submanifold of dimension  $n - 1$ . If we will assume that differentiable function has no critical points, partition of  $M$  into level surfaces of function is a  $n - 1$ - dimensional foliation (codimension one foliation).Codimension one foliations generated by level surfaces were studied in papers [2],[17],[18], [19],[30], [31], [32], [47], [57]. The following theorem gives to us a simple example of foliation.

**Theorem 2.1.** *Let  $f : M \rightarrow N$  be a differentiable mapping of the maximum rank, where  $M$  is a smooth manifold of dimension  $n$ ,  $N$  is a smooth manifold of dimension  $m$ , where  $n > m$ . Then for each point  $q \in N$  a level set  $L_q = \{p \in M : f(p) = f(q)\}$  is a manifold of dimension  $n - m$  and partition of  $M$  into connection components of the manifolds  $L_q$  is a  $n - m$ - dimensional foliation.*

Using the condition 3 of definition 2.1 it is easy to establish that there is a differential structure on each leaf such that a leaf is immersed  $k$ -dimensional submanifold of  $M$ , i.e the canonical injection is a immersing map(a map of the maximum rank). Thus on each leaf there are two topology: the topology  $\tau_M$  induced from  $M$  and it's own topology  $\tau_F$  as a submanifold . These two topologies are generally different. The topology  $\tau_F$  is stronger than topology  $\tau_M$ , i.e. each open subset of  $L_\alpha$  in topology  $\tau_M$  is open in  $\tau_F$ .

A leaf  $L_\alpha$  is called compact if  $(L_\alpha, \tau_F)$  is compact topological space. It is obvious that the compact leaf is a compact subset of manifold  $M$  . The leaf is  $L_\alpha$  called as proper if the topology  $\tau_F$  coincides with the topology  $\tau_M$  induced from  $M$  .If these two these topology on  $L_\alpha$  do not coincide, the leaf is called a non- proper leaf. It is easy to prove that the compact leaf is proper leaf. In work [1] the following assertion is proved which takes place for foliations with singularities too which we will discuss in the second part of this paper.

**Proposition 2.1.** *If a leaf is a closed subset of  $M$  then it is a proper leaf.*

Let  $L$  be a leaf of  $F$ . Point  $y \in M$  is called a limit point of the leaf  $L$  if there is a sequence of points  $y_m$  from  $L$  which converges to  $y$  in topology of manifold  $M$  and does not converge to this point in the topology of the leaf  $L$  [36].

The set of all limit points of the leaf  $L$  we will denote by  $\Omega(L)$ . It is easy to show that the limit set consists of the whole leaves, i.e. if  $y \in \Omega(L)$  then  $L(y) \subset \Omega(L)$ , where  $L(y)$  is a leaf containing  $y$ . Generally the set  $\Omega(L)$  can be empty or can coincide with all manifold. It can already take place for trajectories of dynamic systems. For example, if  $L$  is closed the set  $\Omega(L)$  is empty, and in case of an irrational winding of torus each trajectory everywhere is dense and consequently its limit set coincides with all torus. Studying of limit sets of leaves of foliation includes studying of limit sets of trajectories of the differential equations and it is the important problem of the foliation theory. On these subjects there are numerous researches [1], [14], [25], [33]- [39]. In work [1] following properties of a leaf are proved which also takes place for foliation with singularities too.

**Theorem 2.2.** (1). A leaf  $L_0$  is proper leaf if and only if  $L_0 \cap \Omega(L_0) = \emptyset$  ;

(2). A leaf  $L_0$  is not proper leaf if and only if  $\Omega(L_0) = \bar{L}_0$  where  $\bar{L}_0$  is the closure of  $L_0$  in manifold  $M$ .

For two leaves  $L_1$  and  $L_2$  we will write in  $L_1 \leq L_2$  only in a case when  $L_1 \subset \Omega(L_2)$ . The inequality  $L_1 < L_2$  means  $L_1 \leq L_2$  and  $L_1 \neq L_2$ . The relation  $\leq$  on the set of leaves has been entered by the Japanese mathematician T. Nisimori in the paper [41].

We will denote by  $(M/F, \leq)$  set of leaves with the entered relation on it. It is obvious that the  $\leq$  on  $M/F$  reflective and is transitive, but in many cases this relation is not asymmetric, therefore generally the set  $(M/F, \leq)$  is not partially ordered. T. Nisimori was interested in the case where  $(M/F, \leq)$  is a partially ordered set. Except that T. Nisimori has entered concepts of depth of a leaf  $L$  and depth of foliation  $F$  as follows:  $d(L) = \sup\{k : \text{there exist leaves } L_1, L_2, \dots, L_k \text{ such that } L_1 < L_2 < \dots < L_k = L\}$ ,  $d(F) = \sup\{d(L) : L \in M/F\}$ .

A leaf  $L$ , being the closed subset, has the depth equal to one. It is easy to construct one dimensional foliation of Euclid plane with leaves of depth equal to two.

In work of [41] Nishimori has proved the following theorem which shows that for each positive integer  $k$  there exists two-dimensional foliation with leaves of the depth equal to  $k$ .

**Theorem 2.3.** Let  $S_2$  be a closed surface of a genus 2. For all positive integer  $k$  there is a codimension one foliation  $F$  on  $M = S_2 \times [0, 1]$  satisfying the following conditions (1), (2) and (3).

(1) All leaves of  $F$  are proper and transverse to  $x \times [0, 1]$  for all  $x \in S_2$ .  $S_2 \times 0$  and  $S_2 \times 1$  are compact leaves.

(2)  $d(F) = k$ .

(3) All holonomy groups of  $F$  are abelian.

The following theorem is proved in paper [1] shows that there exists one dimensional analytical foliation generated by integral curves of analytical vector field which have leaves of depth equal to 1, 2 and 3.

**Theorem 2.4.** Let  $S^k$  be a  $k$  dimensional sphere. On the manifold  $S^2 \times S^1$  there exists an analytical vector field without singular points and with three pairwise different integral curves  $\alpha, \beta, \gamma$  such that  $\alpha \subset \Omega(\beta), \beta \subset \Omega(\gamma)$ , where  $\alpha$  is a closed trajectory,  $\Omega(\beta)$  consists of only closed trajectories,  $\Omega(\gamma)$  consists of only the trajectories of depth equal to two.

This vector field generates one-dimensional foliation of the depth equal to 3.

**Remark 2.1.** The example of not analytical dynamic system of a class  $C^\infty$  was constructed in the paper [25] for which there is an infinite chain of not closed trajectories  $L_i$  such that each trajectory  $L_{i+1}$  is in the limit set of  $L_i$ .

In the paper [41] for codimension one foliation the following theorem is proved:

**Theorem 2.5.** (Nishimori). *If  $d(F) < \infty$  or all leaves of foliation  $F$  are proper, then the set  $(M/F, \leq)$  is partially ordered.*

Nishimori, studying property codimension one foliation in the case when the set  $(M/F, \leq)$  is a partially ordered, has delivered following questions which are of interest for foliation with singularities too [41]:

1. Are all leaves of foliation  $F$  proper under the assumption that the set  $(M/F, \leq)$  is partially ordered?
2. Is a leaf  $L$  proper under the assumption that  $dL < \infty$ ?

A.Narmanov studied the relation  $\leq$  for foliation with singularities in the paper [33]. In particular, he proved the following theorems which solves problems 1, 2 delivered by Nishimori.

**Theorem 2.6.** *Let  $M/F$  be the set of leaves of foliation  $F$  with singularities. Then the set  $(M/F, \leq)$  is a partially ordered if and only if all leaves are proper.*

**Theorem 2.7.** *If the depth of a leaf is finite, then it is proper leaf.*

It is known that the limit leaf of compact leaves codimension one foliation on compact manifold is a compact leaf and the limit set of each leaf contains finite number of compact leaves.

The following theorems are generalizations of these facts for leaves with finite depth [34].

**Theorem 2.8.** *Let  $F$  be a transversely oriented codimension one foliation on compact manifold  $M$ ,  $L_i$ - a leaf of foliation  $F$ , and  $x_i \rightarrow x$ , where  $x_i \in L_i$ . If  $dL_i \leq k$  for each  $i$  then  $dL(x) \leq k$ .*

**Theorem 2.9.** *Let  $F$  is a transversely oriented codimension one foliation on compact manifold  $M$ ,  $L^0$  - some leaf of foliation  $F$ . Then for each  $k \geq 1$  the set  $C_k = \{L : L < L^0, dL = k\}$  either is empty, or consists of finite number of leaves.*

Let's remind that transversally orientability of  $F$  means that there exists smooth non-degenerated vector field  $X$  on  $M$ , which is transversal to leaves of foliation  $F$ .

Let  $x \in M$ ,  $L(x)$  is a leaf foliation, containing the point  $x$ ,  $T_x$  is a manifold dimension of  $n - k$  transversal to  $L(x)$  such that  $T_x \cap L(x) = x$ . To each the closed continuous curve in  $L(x)$  beginning and the ending at the point  $x \in M$  corresponds a local diffeomorphism  $g$  of the manifold  $T_x$ , given in some neighborhood of the point  $x$  in  $T_x$  such that  $g(x) = x$ . The set of such diffeomorphisms forms the pseudogroup  $\Gamma_x(L)$  of the leaf  $L$  at the point  $x$ , and germs of these diffeomorphisms form holonomy group  $H$  of the leaf  $L(x)$ . For different points from  $L$  corresponding holonomy groups are isomorphic [53].

The important results in foliation theory are received by G. Reeb. One of his theorems is called as the theorem of local stability which can be formulated as follows.

**Theorem 2.10.** [53] *Let  $L_0$  be a compact leaf foliation  $F$  with finite holonomy group. Then there is an open saturated set  $V$  which contains  $L_0$  and consists of compact leaves.*

Let's notice that a saturated set  $S \subset M$  on a foliated manifold is a subset which is the union of leaves.

In 1976 in Rio de Janeiro at the international conference the attention to the question on possibility of the proof of theorems on local stability for noncompact leaves [50] has been brought. In 1977 the Japanese mathematician T. Inaba has constructed a counterexample which shows that if codimension of foliation is not equal to one G. Reeb's theorem cannot be generalized for noncompact leaves [14].

Let's bring the theorem on a neighborhood of a leaf with finite depth which is generalization of the theorem of J. Reeb on local stability for transversely oriented codimension one foliation.

Let  $F$  transversely oriented,  $X$  a smooth vector field on  $M$ , transversely to leaves of  $F$ . Let  $x \in M, t \rightarrow X^t(x)$ , - the integral curve of a vector field  $X$  passing through the point  $x$  at  $t = 0$ . Let's put  $T_x = \{X^t(x) : -a < t < a\}$ . In further will write  $T_x \approx (-a, a)$  and as usually, to replace subsets of  $T_x$  by their images from  $(-a, a)$ . The point is  $y \in T_x$  called as a motionless

point of pseudo-group  $\Gamma = \Gamma_x(L)$ , if  $g(y) = y$  for each  $g \in \Gamma$ , advanced in a point  $y \in T_x$ . If there exists  $\varepsilon > 0$  such that each point from  $(-\varepsilon, \varepsilon)$  is a motionless point of pseudo-group  $\Gamma$  we will say that the pseudo-group is  $\Gamma$  trivial.

Let  $F$  be a codimension one foliation,  $L$  be a some leaf of  $F$  with finite depth,  $\rho$  - distance function defined by some fixed Riemannian metric on  $M$ .

Let's enter set  $U_r = \{y \in M : \rho(y, L) < r\}$ ,  $r > 0$ , where  $\rho(y, L)$ - distance from the point  $y$  to the leaf  $L$ .

**Theorem 2.11.** ([35]). *Let  $F$  be a transversely oriented codimension one foliation on compact manifold  $M$ . If the holonomy pseudogroup  $\Gamma$  the leaf  $L$  is trivial, then for each  $r > 0$  there is an invariant open set  $V$  containing  $L$  and consisting of leaves diffeomorphic to  $L$  which satisfies to following conditions:*

- 1)  $V \subset U_r$ ;
- 2)  $dL_\alpha = dL$  for each leaf  $L_\alpha \subset V$ .

One more generalization of G. Reeb theorem for a noncompact leaf is resulted below. For this purpose we will bring some definitions.

Let  $M$  be smooth connected complete Riemannian manifold of dimension  $n$  with Riemannian metric  $g$ ,  $F$  -smooth foliation of dimensions  $k$  on  $M$ .

Let's denote through  $L(p)$  a leaf of  $F$  passing through a point  $p$ ,  $F(p)$ - tangent space of leaf at the point  $p$ ,  $H(p)$  - orthogonal complementary of  $F(p)$  in  $T_pM$ ,  $p \in M$ . There are two subbundle (smooth distributions),  $TF = \{F(p) : p \in M\}$ ,  $H = \{H(p) : p \in M\}$  of tangent bundle  $TM$  such, that  $TM = TF \oplus H$  where is  $F$  orthogonal addition  $TF$ .

Piecewise smooth curve  $\gamma : [0, 1] \rightarrow M$  we name horizontal, if  $\dot{\gamma}(t) \in H(\gamma(t))$  for each  $t \in [0, 1]$ . Piecewise smooth curve which lies in a leaf foliation  $F$  is called as vertical.

Let  $I = [0, 1]$ ,  $\nu : I \rightarrow M$  a vertical curve,  $h : I \rightarrow M$  a horizontal curve and  $h(0) = \nu(0)$ . Piecewise smooth mapping  $P : I \times I \rightarrow M$  is called as vertical-horizontal homotopy for pair  $\nu, h$  if  $t \rightarrow P(t, s)$  is a vertical curve for each  $s \in I$ ,  $s \rightarrow P(t, s)$  is a horizontal curve for each  $t \in I$ , and  $P(t, 0) = \nu(t)$  for  $t \in I$ ,  $P(0, s) = h(s)$  for  $s \in I$ . If for each pair of vertical and horizontal curves  $\nu, h : I \rightarrow M$  with  $h(0) = \nu(0)$  there exists corresponding vertical-horizontal homotopy  $P$ , we say that distribution  $H$  is Ehresman connection for  $F$  [3].

Let  $L_0$  a leaf of codimension one foliation  $F$ ,  $U_r = \{x \in M : \rho(x; L_0) < r\}$ , where  $\rho(x, L_0)$ - distance from the point  $x$  to a leaf  $L_0$ . We will assume that there is such number  $r_0 > 0$  that for each horizontal curve  $h : [0, 1] \rightarrow U_{r_0}$  and for each vertical curve  $\nu : [0, 1] \rightarrow L_0$  such that  $h(0) = \nu(0)$  there exists vertically-horizontal homotopy for pair  $(\nu, h)$ . At this assumption we formulate generalization of the theorem of J. Reeb [36].

**Theorem 2.12.** *Let  $F$  be a transversely oriented codimension one foliation,  $L_0$  be a relatively compact proper leaf with finitely generated fundamental group. Then if holonomy group of the leaf  $L_0$  is trivial then for each  $r > 0$  there is an saturated set  $V$  such that  $L_0 \subset V \subset U_r$  and restriction of  $F$  on  $V$  is a fibration over  $R^1$  with the leaf  $L_0$ .*

From the geometrical point of view, the important classes of foliation are total geodesic and Riemannian foliations. Foliation  $F$  on Riemannian manifold  $M$  is called total geodesic if each leaf of foliation  $F$  is a total geodesic submanifold, i.e every geodesic tangent to a leaf foliation  $F$  at one point, remains on this leaf. The geometry of total geodesic foliations is studied in works [13], [16], [38], [4].

Foliation  $F$  on a Riemannian manifold  $M$  is called Riemannian if each geodesic, orthogonal at some point to a leaf of foliation  $F$ , remains orthogonal at all points to leaves of  $F$  [47]. Riemannian foliation without singularities for the first time have been entered and studied by Reinhart in work [47]. This class foliation naturally arising at studying of bundles and level surfaces. Riemannian foliation are studied by many mathematicians, in particular, in works of R. Herman [10], [11], [12], P. Molino [26], A. Morgan [27], Ph. Tondeur [57]. The most simple examples of Riemannian foliation are partition of  $R^n$  into parallel planes or into concentric

hyperspheres. Riemannian foliation with singularities have been entered and studied in works of P. Molino [26], they also were studied by A. Narmanov in the papers [38], [40].

Let  $M$  smooth connected complete Riemannian reducible manifold. Then on  $M$  there are two parallel foliations, mutually additional on orthogonality [20]. If  $M$  simple connected manifold then the de Rham theorem takes place which asserts that  $M$  is isometric to direct product of any two leaves from different foliations [20]. In this case both foliations are Riemannian and total geodesic simultaneously. Below it is presented results of authors on geometry of Riemannian and totally geodesic foliations.

Assume that  $F$  is a Riemannian foliation with respect to Riemannian metric  $g$  on  $M$ . Let  $\pi_1 : TM \rightarrow TF$ ,  $\pi_2 : TM \rightarrow H$  be orthogonal projections,  $V(M)$ ,  $V(F)$ ,  $V(H)$  be the sets of smooth sections of bundles  $TM$ ,  $TF$  and  $H$  accordingly. If  $X \in V(F)$  ( $X \in V(H)$ )  $X$  is a called a vertical (horizontal) field.

Now we will assume that each leaf of  $F$  is a total geodesic submanifolds of  $M$ . It is equivalent to that,  $\nabla_X Y \in V(F)$  for all  $X, Y \in V(F)$  [47], where  $\nabla$ - Levi-Civita connection. In this case  $F$  is a Riemannian foliation with total geodesic leaves. Then on bundles  $TF$  and  $H$  are given metric connections  $\nabla^1$  and  $\nabla^2$  as follows. If  $X \in V(F), Y \in V(H), Z \in V(M)$ , we will put

$$\nabla_Z^1 = \pi_1(\nabla_Z(X)), \nabla_Z^2(Y) = \pi_2[Z_1, \tilde{Y}] + \pi_2[\nabla_{Z_2}, Y],$$

where  $Z = Z_1 \oplus Z_2, Z_1 \in V(F), Z_2 \in V(H), \tilde{Y} \in V(M), \pi_2 \tilde{Y} = Y$ . Here  $[Z_1, \tilde{Y}]$  - Lie bracket of vector fields  $Z_1$  and  $\tilde{Y}$ .  $\nabla_2$  is a metric connection only in case when  $F$  is Riemannian. Owing to that  $F$  is a totally geodesic foliation, connection  $\nabla^1$  also is metric [38], [47].

Let  $p \in M$ ,  $S(p)$  be the set of points  $M$ , which can be connected by horizontal curves with  $p$ . Owing to that foliation  $F$  is total geodesic, for each the  $p \in M$  set  $S(p)$  has topology and differentiable structure, in relation to which  $S(p)$  is a immersed submanifold of  $M$  [4]. It is easy to prove the following assertion.

**Lemma 2.1.** *dim* $S(p) \geq k$  for every point  $p \in M$ .

Owing to that the manifold  $M$  is complete, and considered foliation is a Riemannian, the distribution  $H$  is a Ehresmann connection for  $F$  [3]. Therefore for each piece-wise smooth curve  $\gamma : I \rightarrow M$  there exists unique vertical-horizontal homotopy  $P_\gamma : I \times I \rightarrow M$  such that  $\gamma(t) = P_\gamma(t, t)$  for  $t \in I$ . Let  $P : I \times I \rightarrow M$  be a vertical-horizontal homotopy. We will denote by  $D_t(P(t, s))$  the tangent vector of the curve  $t \rightarrow P(t, s)$  at the point  $P(t, s)$ , by  $D_s(P(t, s))$  the tangent vector of the curve  $s \rightarrow P(t, s)$  at the point  $P(t, s)$ .

**Lemma 2.2.** *Let*  $X(t, s) = D_s(P(t, s)), Y(t, s) = D_t(P(t, s))$  *for*  $(t, s) \in I \times I$ . *Then*  $\nabla_X^1 Y = 0$  *and*  $\nabla_Y^2 X = 0$ .

In the proof of the de Rham theorem projections of a curve  $\gamma$  in  $L(p_0)$  and in  $S(p_0)$  are defined with connection  $\nabla$  and it is shown that these projections coincide with curves  $P_\gamma(\cdot, 0) : I \rightarrow M$  and  $P_\gamma(0, \cdot) : I \rightarrow M$  accordingly. In this case it is used that distribution  $H$  is complete integrable [20]. In the paper [38] the similar fact is proved for projections of a curve  $\gamma$  without the assumption that  $H$  is complete integrable by means of metric connection  $\tilde{\nabla}$  which is entered below. Metric connection  $\tilde{\nabla}$  is defined as follows:

$$\tilde{\nabla}_Z X = \nabla_Z^1 X_1 + \nabla_Z^2 X_2,$$

where  $X, Z \in V(M)$ ,  $X_i = \pi_i(X)$ ,  $i = 1, 2$ . It is not difficult to check up that distributions  $TF$  and  $H$  are parallel with respect to  $\tilde{\nabla}$ .

Let  $\gamma : I \rightarrow M$  be a smooth curve,  $\gamma(0) = p_0$  and  $\gamma(1) = p$ ,  $C : I \rightarrow T_{p_0}M$  is a development of the curve  $\gamma$  in  $T_{p_0}M$  defined by connection  $\tilde{\nabla}$ . (See definition of development in [[20],p.129]. (Here for convenience tangent vector space  $T_{p_0}M$  is identified with affine tangent space at the point  $p_0$ .)

Let  $C(t) = (A(t), B(t))$  where  $A(t) \in F(p_0), B(t) \in H(p_0)$  for  $t \in I$ . As  $M$  is a complete Riemannian manifold,  $\tilde{\nabla}$ - metric connection, there are smooth curves  $\gamma_1, \gamma_2 : I \rightarrow M$ , which are developed on curves  $t \rightarrow A(t)$  and  $t \rightarrow B(t)$  accordingly ([20], p.167, the theorem 4.1). According to the proposition 4.1 in ([20],p.129)  $\gamma_i$  is a such curve that result of parallel transport  $\dot{\gamma}_i$  to the point  $p_0$  along  $\gamma_i^{-1}$  defined by connection  $\tilde{\nabla}$  coincides with the result of parallel transport of  $\pi_i(\dot{\gamma}(t))$  to the point  $p_0$  along  $\gamma^{-1}$  defined by connection  $\tilde{\nabla}$  too. That is why  $\gamma_1$  is a vertical curve,  $\gamma_2$  is a horizontal curve. Curves  $\gamma_1, \gamma_2$  are called projections of the curve  $\gamma$  in  $L(p_0)$  and in  $S(p_0)$  accordingly.

The following theorems are proved in the work [38].

**Theorem 2.13.** *The projection of curve  $\gamma : I \rightarrow V$  in  $L(q_0)$  (in  $S(q_0)$ ) is a curve  $P(\cdot, 0) : I \rightarrow L(q_0)$  (accordingly  $P(0, \cdot) : I \rightarrow S(q_0)$ ).*

The following theorem shows that if distribution  $H$  is complete integrable if and only if connection  $\tilde{\nabla}$  coincides with Levi-Civita connection  $\nabla$ .

**Theorem 2.14.** *Following assertions are equivalent.*

1. *Distribution  $H$  is complete integrable (i.e  $\dim S(p) = n - k$  for each  $p \in M$ ).*
2.  *$\tilde{\nabla}$  is connection without torsion (i.e  $\tilde{\nabla} = \nabla$ ).*

**Remark 2.2.** *As shows known Hopf fibration of on three-dimensional sphere, the distribution  $H$  it is not always complete integrable.*

In a case when  $H$  is complete integrable de Rham theorem takes place: if  $M$  is simple connected, it is isometric to product  $L(p) \times S(p)$  for each  $p \in M$ . In this case  $S(p)$  is a leaf of foliation  $F^\perp$  generated by distribution  $H$ . Projections of any point  $p \in M$  in  $L(p_0)$  and in  $S(p_0)$  are defined as follows. Let  $\gamma : I \rightarrow M$  is a smooth curve,  $\gamma(0) = p_0, \gamma(1) = p, \nu, h$  are projections of  $\gamma$  in  $L(p_0)$  and in  $S(p_0)$ . The points  $\nu(1), h(1)$  are called as projections of  $p$  in  $L(p_0)$  and in  $S(p_0)$  accordingly. Owing to that the distribution  $H$  is completely integrable, the projection of  $p$  depends only on the homotopy class of the curve  $\gamma$ . That is why when  $M$  is simple connected, the mapping  $f : p \rightarrow (p_1, p_2)$  is correctly defined. Under theorems 1.12 and 1.13 mapping  $f$  is a isometric immersing. Since  $\dim M = \dim\{L(p_0) \times S(p_0)\}$  the mapping  $f$  is covering mapping, hence, it is an isometry ([20], p.134).

In the known monograph [57] Ph. Tondeur studied foliation, generated by level surfaces of functions of a certain class. He considered function  $f : M \rightarrow R^1$  without critical points on Riemannian manifold  $M$  for which length of a gradient is constant on each level surface. For such functions he has proven that foliation generated by level surfaces of such function, is a Riemannian foliation. Authors of the present article studied geometry of foliation generated by level surfaces of the functions considered in the monograph of professor Ph. Tondeur without the assumption of absence of critical points.

**Definition 2.2.** *Let  $M$  be a smooth manifold of dimension  $n$ . Function  $f : M \rightarrow R^1$  of the class  $C^2(M, R^1)$  for which length of a gradient is constant on connection components of level sets is called a metric function.*

Let  $f : R^n \rightarrow R^1$  be a metric function. We will consider system of the differential equations

$$\dot{x} = \text{grad}f(x). \quad (1)$$

As, the right part of system 1 is differentiable, for each point  $x_0 \in M$  there is a unique solution of system 1 with the initial condition  $x(0) = x_0$ . The trajectory of system 1 is called gradient line of the function  $f$ .

**Theorem 2.15.** [17]. *Curvature of each gradient line of metric function is equal to zero.*

In the papers [17], [19] topology of level surfaces are studied under the assumption that each of a connection components of the set of critical points of metric function is either a point, or is a regular surface and every component is isolated from others.

The following theorem gives complete classification of foliations generated by level surfaces of metric function [19].

**Theorem 2.16.** *Let  $f : M \rightarrow R^1$  be a metric function given in  $R^n$ . Then level surfaces of function  $f$  form foliation which has one of following  $n$  types :*

- 1) *Foliation  $F$  consists of parallel hyperplanes;*
- 2) *Foliation  $F$  consists of concentric hyperspheres and the point (the center of hyperspheres);*
- 3) *Foliation  $F$  consists of concentric cylinders of the kind  $S^{n-k-1} \times R^k$  and the singular leaf  $R^k$  (which arises at degeneration of spheres to a point), where  $k$ - the minimum of dimensions of critical level surfaces,  $1 \leq k \leq n - 2$ .*

At the proof of the 2.1 4 the following theorem is used which also is proved in work [19].

**Theorem 2.17.** *Let  $L$  be a regular surface of dimension  $r$  which is the closed subset  $R^n$ , where  $1 \leq r \leq n - 1$ . If the normal planes passing through various points  $L$  are not crossed, then  $L$  is  $r$ - a dimensional plane.*

This theorem represents independent interest for the course of differential geometry.

The following theorem shows internal link between geometry of Riemannian manifold and property of metric function which is given on it.

**Theorem 2.18.** [32] *Let  $(M, g)$  be a smooth Riemannian manifold of dimension  $n$ ,  $f : M \rightarrow R^1$  metric function. Then each gradient line of the function  $f$  is a geodesic line of the Riemannian manifold  $M$ .*

For the metric functions given on a Riemannian manifold , it is difficult to get classification theorems, as in a case  $M = R^n$ . Here one much depends on topology of Riemannian manifold on which function is given. For example, in Euclid case if metric function has no critical points, then as shown [19] all level surfaces are hyperplanes. It is easy to construct metric function without critical points on the two-dimensional cylinder ( with the metric induced from Euclid structure of  $R^3$ ) level lines of which are circles (compact sets).

The following theorem, is the classification theorem for level lines of the metric function given on two-dimensional Riemannian manifold.

**Theorem 2.19.** [32] *Let  $M$  be two-dimensional Riemannian manifold,  $f : M \rightarrow R^1$  be a metric function without critical points. Then all level lines are homeomorphic to circle, or all level lines are homeomorphic to a straight line.*

The following theorem shows that if the metric function is given on complete simple connected Riemannian manifold and does not have critical points, then it has no compact level surfaces.

**Theorem 2.20.** [32] *Let  $M$  be a smooth complete simple connected Riemannian manifold,  $f : M \rightarrow R^1$  be a metric function without critical points. Then level surfaces are mutually diffeomorphic noncompact submanifolds of  $M$ .*

### 3. APPLICATIONS OF FOLIATION THEORY IN CONTROL SYSTEMS

The last years methods and results of foliation theory began to be used widely in the qualitative theory of optimal control. It was promoted by works of the American mathematician of G. Sussmann [52] and the English mathematician P. Stefan [51] which have shown that a orbit of family of smooth vector fields is a smooth immersed submanifold. Besides, they have shown that if dimensions of orbits are the same, partition of phase space into orbits is a foliation. Papers [33], [35], [36], [37], [7], [39], [40], [43], [44] are devoted to applications of foliation theory in control theory.

Let  $D$  be a family of the smooth vector fields on  $M$ , and  $X \in D$ . Then for a point  $x \in M$  by  $X^t(x)$  we will denote the integral curve of a vector field  $X$  passing through the point  $x$  at  $t = 0$ . Mapping  $t \rightarrow X^t(x)$  is defined in some domain  $I(x)$  which generally depends not only on a field



$X$ , but also from the point  $x$ . Further everywhere in formulas kind of  $X^t(x)$  we will consider that  $t \in I(x)$ . The orbit  $L(x)$  passing through a point  $x$  of the family of the vector fields  $D$  is defined as a set of points  $y$  from  $M$  for which there are real numbers  $t_1, t_2, \dots, t_k$  and vector fields  $X_{i_1}, X_{i_2}, \dots, X_{i_k}$  from (where  $k$  is a natural number) such that  $y = X_{i_k}^{t_k}(X_{i_{k-1}}^{t_{k-1}}(\dots(X_{i_1}^{t_1})\dots))$ .

Now we will bring a definition of foliation with singularities [51]. A subset  $L$  of manifold  $M$  is called as a leaf if

1) there is a differential structure  $\sigma$  on  $L$  such that  $(L, \sigma)$  is a connected  $k$ - dimensional immersed submanifold of  $M$ .

2) for locally connected topological space  $N$  and for continuous mapping  $f : N \rightarrow M$  such that  $f(N) \subset L$  the mapping  $f : N \rightarrow (L, \sigma)$  is continuous.

Partition  $F$  of manifold  $M$  into leaves is called smooth (of the class  $C^r$ ) foliation with singularities if following conditions are satisfied:

1) For each point  $x \in M$  there exists a local  $C^r$ - chart  $(\psi, U)$  containing the point  $x$  such that  $\psi(U) = V_1 \times V_2$  where  $V_1$  is a neighborhood of origin in  $R^k$ ,  $V_2$ - a neighborhood of origin in  $R^{n-k}$ ,  $k$ - dimension of the leaf containing the point  $x$ ;

2)  $\psi(x) = (0, 0)$  ;

3) For each leaf  $L$  such that  $L \cap U \neq \emptyset$  it takes place equality  $L \cap U = \psi^{-1}(V_1 \times l)$  where  $l = \{\nu \in V_2 : \psi^{-1}(0, \nu) \in L\}$ .

By definition 2.1 each regular foliation is a foliation in sense of the above-stated definition. In this case every connection component of the set  $l$  is a point. If dimensions of leaves of a foliation with singularities are the same as noted above, it is a foliation in sense of the definition 2.1. Thus, the conception of foliation with singularities is a generalization of classical notion of a foliation (now which is called as regular foliation). In the literature instead of "foliation with singularities" the term "singular foliation" is used also [26]. To the studying of a foliation with singularities are devoted papers [1], [26], [36], [37], [51].

Now we will consider some applications of the foliation theory in problems of the qualitative theory of control systems.

Let's consider a control system

$$\dot{x} = f(x, u), x \in M, u \in U \subset R^m, \quad (2)$$

where  $M$  is a smooth (class  $C^\infty$ ) connected manifold of dimension  $n$  with some Riemannian metric  $g$ ,  $U$  is a compact set, for each the  $u \in U$  vector field  $f(x, u)$  is a field of class  $C^\infty$ , and mapping  $f : M \times U \rightarrow TM$ , where  $TM$  is the tangent bundle of  $M$ , is continuously differentiable. It means that there is such open set  $V$  such that  $U \subset V \subset R^m$ , and continuously differentiable mapping  $\bar{f} : M \times V \rightarrow TM$ , restriction of which on  $M \times U$  coincides with  $f(x, u)$ . Admissible controls are defined as piecewise-constant functions  $u : [0, T] \rightarrow U$ , where  $0 < T < \infty$ . Thus, the trajectories of system 2 corresponding to admissible controls, represent piecewise smooth mapping  $x : [0, T] \rightarrow M$ .

The purpose of control is a bring of the system to some fixed (target) point  $\eta \in M$ . We will say that the point  $x_0 \in M$  is controllable from a point  $\eta$  in time  $T > 0$ , if there is such trajectory of  $x : [0, T] \rightarrow M$  of the system 2 that  $x(0) = x_0, x(T) = \eta$ . Let's denote by  $G_\eta(< T)$  a set of points of  $M$  which are controllable from a point  $\eta$  for time, smaller than  $T$ . We assume that  $\eta \in G_\eta(< T)$  for each  $T > 0$ . The set of all points  $M$ , which are controllable from a point  $\eta$ , is called as set of controllability with a target point  $\eta$  and is denoted by  $G_\eta$ . We will denote by  $T = T_\eta(x)$  function of Bellman given on set  $G_\eta$  for the optimal time problem. It is known that a set of smooth vector fields on a smooth manifold can be transformed into Lie algebra in which as product of vector fields  $X$  and  $Y$  serves their Lie bracket  $[X, Y]$ .

Let's denote by  $D$  set of vector fields  $\{f(\cdot, u) : u\} \in U$ , by  $A(D)$  minimal Lie subalgebra, containing  $D$ , by  $A_x(D)$  the subspace tangent spaces at a point  $x \in M$ , consisting of all vectors  $\{X(x) : X \in a(D)\}$ . If we will denote by  $L(\eta)$  a orbit of family  $D$  containing the point  $\eta$ , then

it follows from definition of the orbit that  $G_\eta \subset L(\eta)$  for all  $\eta \in M$ . The following assertion [43] takes place.

**Theorem 3.1.** *If  $\dim A_\eta(D) = \dim L(\eta)$  then  $\text{int}G_\eta \neq \emptyset$  in topology of  $L(\eta)$ .*

Now let us give following definitions.

**Definition 3.1.** *We will say that the system 2 is completely controllable on  $L(\eta_0)$ , if for all  $\eta \in L(\eta_0)$  it takes place equality  $G_\eta = L(\eta_0)$  .*

**Definition 3.2.** *The system 2 is called normally -locally controllable (or, more shortly,  $N$ -locally controllable) near a point  $\eta$  if for any  $T > 0$  there is a neighborhood  $V$  of the point  $\eta$  in  $L(\eta)$  such that each point from  $V$  is controllable from  $\eta$  in time, smaller  $T$ .*

If the system is  $N$ - locally controllable near each point of  $L(\eta)$  we will say that it is  $N$  - locally controllable on  $L(\eta)$  (see [52]).

**Definition 3.3.** *We will say that the system 2 is completely ( $N$  -locally) controllable on invariant set  $S$  if it is completely controllable (  $N$  - locally controllable) on each a leaf of  $S$ .*

Assume that  $\dim A_x(D) = k$  for every  $x \in M$ , where  $0 < k < n$ ,  $A_x(D) = \{X(x) : X \in A(D)\}$ . In this case splitting of  $F$  manifold  $M$  into orbits family  $D$  is a  $k$ -dimensional foliation, i.e.orbits are  $k$  dimensional submanifolds of  $M$ .

Let's consider the following question: if the system 2 has property of complete controllability on one fixed leaf of the foliation  $F$ , under what conditions the system 2 has this property on leaves close to a given leaf?

This question closely related with problems of the qualitative theory of foliations on local stability of a leaf in sense of J.Reeb(see [53]). In the paper [7] the answer is given to this question in the case when the leaf  $L_0$  of  $F$  in neighborhood of which the system 2 is studied,is compact set. In this case conditions of the theorem of J.Reeb on local stability is required. Namely, the following theorem is proved.

**Theorem 3.2.** *Let  $L_0$  be a compact leaf of  $F$  with finite holonomy group.If the system 2 is complete controllable on  $L_0$  then it is complete controllable on leaves,close to  $L_0$ .*

Thus, existence of such saturated (invariant) neighborhood  $V$  of a leaf  $L_0$  gives the sufficient condition for stability of the complete controllable system 2 on  $L_0$  , when  $L_0$  is a compact leaf. As Example 3 in [7] shows, complete controllability on close leaves does not follow from the fact that the system 2 is complete controlled on a noncompact proper leaf  $L_0$  having a neighborhood  $V$  described in theorem 3.1. Therefore, in a case when  $L_0$  is a noncompact leaf, we need additional conditions that guarantee stability of the complete controlled system 2 on  $L_0$ . The theorem 2.12 gives the possibility to get sufficient conditions for stability of the complete controllable system 2 on  $L_0$  , when  $L_0$  is a noncompact leaf.

**Theorem 3.3.** [32] *Let  $F$  be a transversely oriented codimension one foliation,  $L_0$  be a relatively compact proper leaf with finitely generated fundamental group and with trivial holonomy group. Then if the system 2 is  $N$ -locally controllable on  $\bar{L}_0$  (the closure in  $M$ ) then there is an open saturated neighborhood  $V$  of the leaf  $L_0$  such that on each leaf from  $V$  the system 2 is complete controllable.*

**Remark 3.1.** *The closure of each leaf is an invariant set (see ([53], Theorem 4.9).*

**Remark 3.2.** *If the system 2 is  $N$ -locally controlled on a leaf  $L$  then it is complete controlled on  $L$  (see [39]).*

Let now  $\dim A_x(D) = k$  for every  $x \in M$  where  $0 < k < n$ ,  $F$  is a Riemannian foliation with respect to Riemannian metric  $g$ . We will remind that foliation  $F$  is called Riemannian, if each geodesic orthogonal at some point to a leaf foliation  $F$  , remains orthogonal to leaves at all points.

**Theorem 3.4.** [39] *Let  $(M, g)$  be a complete Riemannian manifold,  $L_0$  be a relatively compact proper leaf of  $F$ . Then if the system 2  $N$ -is locally controllable on  $\bar{L}_0$  (on the closure of  $L_0$  in  $M$ ) then there is an invariant neighborhood  $V$  of the leaf  $L_0$  such that system (1) is complete controllable on each leaf from  $V$ .*

In the paper [26] it is given the necessary and sufficient condition for singular foliation  $F$  to be Riemannian foliation. This condition deals with vector fields from  $A(D)$  and Riemannian metric  $g$ .

Let now  $F$  be  $k$ -dimensional Riemannian foliation with respect to Riemannian metric  $g$ , where  $0 < k < n$ . It is possible to present each vector field  $X \in V(M)$  as  $X = X_P + X_H$  where  $X_P, X_H$  orthogonal projections of  $X$  on  $P$  and  $H$  accordingly. If  $X_H = 0$ , then  $X \in V(F)$  and the vector field  $X$  is called tangent vector field, if  $X_P = 0$  then  $X \in V(H)$  and the vector field  $X$  is called horizontal field. For vector fields  $X, Y$  we will consider the bilinear symmetric form  $g_T(X, Y) = g(X_H, Y_H)$  on  $V(M)$ , kernel of which coincides with  $V(F)$ . We will study properties of this form. By the definition of foliation for each point  $p \in M$  there is a neighborhood  $U$  of the point  $p$  and local system of coordinates  $x^1, x^2, \dots, x^k, y^1, y^2, \dots, y^{n-k}$  on  $U$  such that  $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^k}$  form basis of sections of  $TF|_U$ . The basis  $\nu_{k+1}, \nu_{k+2}, \dots, \nu_n$  for sections  $H|_U$  can be chosen in such a manner that brackets  $[X, \nu_j]$  will be tangent vector fields to foliation  $F$  for each section  $X$  of the bundle  $H|_U$ . Now assume that foliation  $F$  is a Riemannian. Then for each tangent vector  $X \in V(F)|_U$  it takes place  $X_g(\nu_i, \nu_j) = 0, i, j = k + 1, \dots, n$  [40]. By using this fact, it is easy to show that for each vector field  $X \in V(F)$  takes place

$$Xg_T(Y, Z) = g_T([X, Y], Z) + g_T([Y, [X, Z]]),$$

where  $Y, Z \in V(M)$ . In this case is  $g_T$  called transversal metrics for foliation  $F$ , defined by the Riemannian metric  $g$  ([26], see p.77). Notice that a transversal metrics  $g_T$  determines the local distance between the leaves, since it defines the length of the perpendicular geodesics. As follows from ([26] the assertion 3.2), it is true also the converse fact i.e. if it is given  $k$ -dimensional foliation  $F$  on Riemannian manifold and Riemannian metric  $g$  defines transversal metric for  $F$  then  $F$  is a Riemannian foliation with respect to Riemannian metric  $g$ . Authors proved the similar fact for foliations with singularities.

Let  $F$  be a foliation with singularities,  $L$  is a leaf of the foliation  $F$ ,  $Q$  is a normal bundle of  $L$ . Then Riemannian metric  $g$  defines the metric  $g_T^L$  on  $Q$  as follows: if  $\nu_1, \nu_2 : L \rightarrow Q$  are smooth (of the class  $C^\infty$ ) sections of normal bundle  $Q$  we will put  $g_T^L(\nu_1, \nu_2) = g(X, Y)$ , where  $X, Y \in V(M)$ , the restrictions of  $X, Y$  on  $L$  coincides with  $\nu_1, \nu_2$  accordingly. The metric  $g_T^L$  is called transversal metric for  $F$  on  $L$ , if for each  $X \in V(F)$  at points of the leaf  $L$  takes place

$$Xg_T^L(Y, Z) = g_T^L([X, Y], Z) + g_T^L([Y, [X, Z]]),$$

where  $Y, Z \in V(M)$ ,  $g_T^L(Y, Z) = g(\pi Y, \pi Z)$ ,  $\pi : TM \rightarrow Q$  is the orthogonal projection considered over  $L$ . We will notice that Riemannian foliation with singularities has no  $n$ -dimensional leaves. There is an assumption in ([26], p. 201) that if complete Riemannian metric defines on each leaf foliation  $F$  transversal metric, then foliation  $F$  will be a Riemannian. This problem is solved positively by following theorem.

**Theorem 3.5.** [40] *Let  $M$  be a complete Riemannian manifold with Riemannian metric  $g$ ,  $F$  is a singular foliation on  $M$  which has no  $n$ - dimensional leaves. Then the foliation  $F$  is a Riemannian if and only if Riemannian metric  $g$  defines on each leaf of the foliation  $F$  transversal metric.*

Now we will assume that  $x_0 \in M$ ,  $L_0 = L(x_0)$  is a proper leaf with trivial holonomy group and the system 2 is completely controllable on  $L_0$ .

**Theorem 3.6.** [1] *Let the mapping  $x \rightarrow L(x)$  be continuous at a point  $x_0$ . Then the system 2 is completely controllable on the orbits, sufficiently close to  $L_0$ .*

**Remark 3.3.** *Multiple-valued mapping  $x \rightarrow L(x)$  is called to be lower semicontinuous at a point  $x_0$  if for each open set  $V$  such that  $V \cap L(x_0) \neq \emptyset$  there exists neighborhood  $B_{x_0}$  of the point  $x_0$  such that  $L(x) \cap V \neq \emptyset$  for  $x \in B_{x_0}$ .*

Multiple-valued mapping  $x \rightarrow L(x)$  is called to be upper semicontinuous at a point  $x_0$  if for each open set  $V$  such that  $L(x_0) \subset V$ , there is a neighborhood  $B_{x_0}$  of the point  $x_0$  such that  $L(x) \subset V$  for all  $x \in B_{x_0}$ . Multiple-valued mapping is continuous at a point  $x_0$  if it simultaneously lower and upper semicontinuous at a point  $x_0$ . In our case is easy to prove that mapping  $x \rightarrow L(x)$  is lower semicontinuous at each point of  $M$  [36].

Sufficient conditions at which mapping  $x \rightarrow L(x)$  is continuous, are studied in papers of A.Narmanov [36], [37]. We will bring some of them. The following sufficient condition on continuity of multiple-valued mapping  $x \rightarrow L(x)$  follows directly from the theorem 2.10 of the part one.

**Theorem 3.7.** *Assume that  $\dim A_x(D) = k$  for every  $x \in M$ , where  $0 < k < n$ . If the set  $L(x_0)$  is a compact leaf with finite holonomy group then mapping  $x \rightarrow L(x)$  is continuous at the point  $x_0$ .*

The following theorem shows that if foliation  $F$  is a singular Riemannian foliation then the mapping  $x \rightarrow L(x)$  is continuous at each point  $x_0$ .

**Theorem 3.8.** ([36]) *Let  $F$  is a Riemannian foliation with singularities. Then multiple-valued mapping  $x \rightarrow L(x)$  is continuous at each point of the manifold  $M$ .*

Now we will consider the problem on a continuity Bellman function for a optimal time problem. We will remind that Bellman function  $T_\eta(x) : G_\eta \rightarrow R^1$  is defined as follows:  $T_\eta(\eta) = 0$ ,  $T_\eta(x) = \inf(\tau : \text{there exists trajectory } \alpha : [0, \tau] \rightarrow M \text{ of the system 2 such that } \alpha(0) = x, \alpha(\tau) = \eta)$ . The structure of set of controllability generally can be rather difficult. Now we will determine a class of control systems for which a set of controllability  $G_\eta$  of the system 2 for all  $\eta \in M$  coincides with a orbit  $L(\eta)$  of family of the vector fields  $D = \{f(\cdot, u) : u \in U\}$ .

**Definition 3.4.** [43] *We will say that the system 2 is continuously-balanced at a point  $x \in M$  if for each vector field  $X \in D$  there are vector fields  $X_1, X_2, \dots, X_k$  from  $D$ , a neighborhood  $V(x)$  of the point  $x$  and the positive continuous functions  $\lambda_1(y), \lambda_2(y), \dots, \lambda_k(y)$  which are given in this neighborhood such that for all  $y \in V(x)$  takes place equality:  $X(y) + \sum \lambda_i(y)X_i(y) = 0$ .*

If we assume that the system 2 is continuously-balanced at each point  $x \in M$  by means of results of work [51] it is possible to show that for each  $\eta \in M$  the set of controllability  $G_\eta$  of system 2 coincides with the orbit  $L(\eta)$  of the family  $D = \{f(\cdot, u) : u \in U\}$ .

**Definition 3.5.** *We will say that function  $T = T_\eta(x)$  is continuous at the point  $x_0 \in G_\eta$  if for every  $\varepsilon > 0$  there is such neighborhood  $V$  of the point  $x_0$  in topology of  $M$  that for any point  $x \in G_\eta \cap V$  takes place inequality  $|T_\eta(x) - T_\eta(x_0)| < \varepsilon$ .*

For the system 2 given on compact manifold the following result is obtained by professor N.N.Petrov [44].

**Theorem 3.9.** *Let  $M$  be compact manifold. Then following assertions are equivalent:*

- 1) *System 2 is  $N$ - locally controllable near the point  $\eta$ .*
- 2) *For each  $T > 0$  the set  $G_\eta(< T)$  is a domain in the manifold  $L(\eta)$ .*
- 3) *For each  $T > 0$  the level  $x \in M : T_\eta(x) = T$  is the border of the set  $G_\eta(< T)$ .*
- 4) *Bellman function  $T = T_\eta(x)$  is continuous at every point of  $G_\eta$ .*

Thus, for compact manifold the problem on a continuity of Bellman function is reduced to a question about  $N$  - local controllability of system 2. For noncompact manifolds the following theorem gives the necessary and sufficient conditions of a continuity of Bellman function which is presented in [44].

**Theorem 3.10.** *Let the system 2 be continuous-balanced at each point of  $M$ , for each  $T > 0$  the set  $G_\eta(\leq T)$  has compact closure and  $\dim A_x(D) = \text{const}$  for every  $x \in G_\eta$ . Then Bellman function is continuous on  $G_\eta$  if and only if  $G_\eta$  is a proper leaf of the foliation generated by orbits of  $D$ .*

In a case when manifold  $M$  is a analytic and the set  $D$  consists of analytical vector fields owing to theorem Nagano [28] the condition  $\dim A_x(D) = \text{const}$  for all  $x \in G_\eta$  is always satisfied. Generally  $\dim A_x(D)$  can vary from a point to a point on  $G_\eta$  and always  $\dim A_x(D) \leq \dim L_\eta$  for  $x \in G_\eta$ . For continuously-balanced control systems the following theorem takes place [37].

**Theorem 3.11.** *The set  $G_\eta$  is a proper leaf of the foliation generated by orbits of the family  $D$  if and only if the set  $G_\eta$  is a set of type  $F_\sigma$  and  $G_\delta$  simultaneously in topology of manifold  $M$ .*

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